6 Isomorphisms

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Recognizing the maps that preserve a given mathematical structure allows one to compare objects and identify equivalent ones.

6.0 Isomorphisms

When are two rings the same? We want to identify equivalent rings.

6.0.0 Definition. A ring homomorphism $\varphi\colon R \to S$ is called a (ring) *isomorphism* if there exists a ring homomorphism $\psi\colon S \to R$ such that $\psi\circ\varphi = \mathrm{id}_R$ and $\varphi\circ\psi = \mathrm{id}_S$. An *automorphism* is an isomorphism that is also an endomorphism.

The words "isomorphism" and "automorphism" come from the Greek words *isos* meaning 'equal', *autós* meaning 'self', and *morphē* meaning 'form' or 'shape'.

6.0.1 Problem. Show that conjugation is an automorphism of the field $\mathbb{C}$ of complex numbers.

Proof. Problem 5.0.1 shows that conjugation is an endomorphism. For every $z = a + b\,\mathrm{i} \in \mathbb{C}$, with $a, b \in \mathbb{R}$, we have

$$\overline{\overline{z}} = \overline{\overline{a + b\,\mathrm{i}}} = \overline{a - b\,\mathrm{i}} = a + b\,\mathrm{i} = z\ .$$

Thus, conjugation is an involution (equal to its own inverse) and thereby an automorphism. □

6.0.2 Proposition. *A ring homomorphism $\varphi\colon R \to S$ is an isomorphism if and only if the map φ is a bijection.*

A bijection is an isomorphism of sets.

Proof. Suppose that the map $\varphi\colon R \to S$ is a ring isomorphism. By definition, there exists a ring homomorphism $\psi\colon S \to R$ such that $\psi\circ\varphi = \mathrm{id}_R$ and $\varphi\circ\psi = \mathrm{id}_S$. In particular, the underlying map of sets has an inverse, so it is a bijection.

Conversely, suppose that the underlying map of sets is a bijection. Hence, there exists a map $\psi\colon S \to R$ of sets such that $\psi \circ \varphi = \mathrm{id}_R$ and $\varphi \circ \psi = \mathrm{id}_S$. It remains to show that ψ is a ring homomorphism. Since the map φ is a ring homomorphism, it follows that, for all $a, b \in S$, we have $\varphi(\psi(a) + \psi(b)) = \varphi(\psi(a)) + \varphi(\psi(b)) = a + b$, $\varphi(\psi(a)\,\psi(b)) = \varphi(\psi(a))\,\varphi(\psi(b)) = a\,b$, and $\varphi(1_R) = 1_S$. Applying ψ to both side of these equations gives

$$\psi(a) + \psi(b) = \psi\Big(\varphi(\psi(a) + \psi(b))\Big) = \psi(a + b)\,,$$
$$\psi(a)\,\psi(b) = \psi\Big(\varphi(\psi(a)\,\psi(b))\Big) = \psi(a\,b)\,,$$

and $1_R = \psi(\varphi(1_R)) = \psi(1_S)$, so ψ is a ring homomorphism. □

6.0.3 Remark. Problem 5.2.6 shows that $\mathbb{Z}\big/\langle 6\rangle \cong \mathbb{Z}\big/\langle 2\rangle \times \mathbb{Z}\big/\langle 3\rangle$.

6.0.4 Problem. Prove that the field $\mathbb{C}$ is isomorphic to the subring

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathrm{Mat}(2, \mathbb{R}) \;\middle|\; a, b \in \mathbb{R} \right\}.$$

Proof. Let $\varphi\colon \mathbb{C} \to \mathrm{Mat}(2, \mathbb{R})$ be the map defined, for all $a, b \in \mathbb{R}$, by

$$\varphi(a + b\,\mathrm{i}) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

For all $a, b, c, d \in \mathbb{R}$, we obtain

$$\varphi\big((a+b\,\mathrm{i})+(c+d\,\mathrm{i})\big) = \varphi\big((a+c)+(b+d)\,\mathrm{i}\big) = \begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \varphi(a+b\,\mathrm{i}) + \varphi(c+d\,\mathrm{i}),$$

$$\varphi\big((a+b\,\mathrm{i})(c+d\,\mathrm{i})\big) = \varphi\big((ac-bd)+(ad+bc)\,\mathrm{i}\big) = \begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \varphi(a+b\,\mathrm{i})\,\varphi(c+d\,\mathrm{i}),$$

$$\varphi(1) = \varphi(1+0\,\mathrm{i}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so the map φ is a ring homomorphism. By construction, the map φ surjects onto the given subring. Since $\operatorname{Ker}(\varphi) = \{0\}$, the map φ is also injective. Therefore, φ is a ring isomorphism. □

6.0.5 Definition. The rings R and S are *isomorphic*, denoted by $R \cong S$, if there exists a ring isomorphism $\varphi : R \to S$.

6.0.6 Proposition. *Being isomorphic is an equivalence relation.*

Proof. Let R, S, and T be rings.

(*reflexive*) The identity map $\mathrm{id}_R : R \to R$ is a ring homomorphism and its own inverse, so id_R is a ring isomorphism and $R \cong R$.

(*symmetric*) Suppose that $R \cong S$. There are ring homomorphisms $\varphi : R \to S$ and $\psi : S \to R$ such that $\psi \circ \varphi = \mathrm{id}_R$ and $\varphi \circ \psi = \mathrm{id}_S$. In particular, the ring isomorphism $\psi : S \to R$ implies that $S \cong R$.

(*transitive*) Suppose that $R \cong S$ and $S \cong T$. By definition, there exist ring isomorphisms $\varphi : R \to S$ and $\theta : S \to T$. As the composition $\theta \circ \varphi$ is a ring homomorphism,

$$\varphi^{-1} \circ \theta^{-1} \circ \theta \circ \varphi = \varphi^{-1} \circ \mathrm{id}_S \circ \varphi = \varphi^{-1} \circ \varphi = \mathrm{id}_R \text{ , and}$$
$$\theta \circ \varphi \circ \varphi^{-1} \circ \theta^{-1} = \theta \circ \mathrm{id}_S \circ \theta^{-1} = \theta \circ \theta^{-1} = \mathrm{id}_T ,$$

the composition $\theta \circ \varphi : R \to T$ is a ring isomorphism and $R \cong T$. □

Ring isomorphisms preserve all ring-theoretic properties.

6.0.7 Proposition. *Let $\varphi : R \to S$ be a ring isomorphism. An element $b \in R$ is a zerodivisor if and only if the image $\varphi(b) \in S$ is a zerodivisor. Similarly, an element $b \in R$ is a unit if and only if $\varphi(b) \in S$ is a unit.*

Proof. Suppose that $b \in R$ is a zerodivisor. There exists $0 \neq a \in R$ such that $a\,b = 0_R$ or $b\,a = 0_R$. The ring homomorphism gives

$$\varphi(a)\,\varphi(b) = \varphi(ab) = \varphi(0_R) = 0_S \text{ or}$$
$$\varphi(b)\,\varphi(a) = \varphi(ba) = \varphi(0_R) = 0_S .$$

Since the map φ is an isomorphism, its injectivity establishes that $\varphi(a) \neq \varphi(0_R) = 0_S$, so the image $\varphi(b) \in S$ is a zerodivisor. Applying the same argument to $\varphi^{-1} : S \to R$ gives the converse implication.

Suppose that $b \in R^\times$. There exists $a \in R$ such that $a\,b = b\,a = 1_R$. Since φ is a ring homomorphism, we have

$$\varphi(a)\,\varphi(b) = \varphi(a\,b) = \varphi(1_R) = 1_S = \varphi(1_R) = \varphi(b\,a) = \varphi(b)\,\varphi(a),$$

so $\varphi(b) \in S^\times$. Again, applying the same argument to $\varphi^{-1} : S \to R$ gives the converse implication. □

6.0.8 Corollary. *Let R and S be isomorphic rings. The ring R is a domain if and only if the ring S is a domain. Similarly, the ring R is a field if and only if the ring S is a field.* □

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6.1 Isomorphism Theorems

What is the most important source of isomorphisms? Formulated by Emmy Noether (1927), the four isomorphism theorems describe universal features of quotient rings, homomorphisms, and subrings.

Although we focus on rings, versions of these theorems exist for many algebraic structures, including groups and vector spaces.

We first observe that a ring homomorphism descends to quotient rings under a mild hypothesis.

6.1.0 Induced Map Lemma. *Let $\varphi : R \to S$ be a ring homomorphism. For each ideal I in R and each ideal J in S satisfying $\varphi(I) \subseteq J$, the induced map $\overline{\varphi} : R/I \to S/J$, defined for all $r \in R$ by $\overline{\varphi}(r + I) = \varphi(r) + J$, is a well-defined ring homomorphism.*

Proof. Consider $r, s \in R$. When $r \sim_I s$, it follows that $r - s \in I$ and $\varphi(r) - \varphi(s) = \varphi(r - s) \in \varphi(I) \subseteq J$. We deduce that $\varphi(r) \sim_J \varphi(s)$ and $\overline{\varphi}(r + I) = \overline{\varphi}(s + I)$, so the map $\overline{\varphi}$ is well defined. Since φ is a ring homomorphism, we also have

$$\begin{aligned}\overline{\varphi}\big((r+I)+(s+I)\big) &= \overline{\varphi}\big((r+s)+I\big) = \varphi(r+s)+J\\ &= \big(\varphi(r)+\varphi(s)\big)+J\\ &= \big(\varphi(r)+J\big)+\big(\varphi(s)+J\big) = \overline{\varphi}(r+I)+\overline{\varphi}(s+I)\,,\\ \overline{\varphi}\big((r+I)(s+I)\big) &= \overline{\varphi}\big((rs)+I\big) = \varphi(rs)+J = \big(\varphi(r)\varphi(s)\big)+J\\ &= \big(\varphi(r)+J\big)\big(\varphi(s)+J\big) = \overline{\varphi}(r+I)\,\overline{\varphi}(s+I)\,,\end{aligned}$$

and $\overline{\varphi}(1_R+I) = \varphi(1_R)+J = 1_S+J$, so $\overline{\varphi}$ is a ring homomorphism. □

The quotient by the kernel is always isomorphic to the image.

6.1.1 First Isomorphism Theorem. *Let $\varphi : R \to S$ be a ring homomorphism with $I := \operatorname{Ker}(\varphi)$. The induced map $\overline{\varphi} : R/I \to \operatorname{Im}(\varphi)$, defined by $\overline{\varphi}(r+I) = \varphi(r)$ for all $r \in R$, is a ring isomorphism. When $\pi : R \to R/I$ is the quotient map and $\iota : \operatorname{Im}(\varphi) \to S$ is the inclusion, we have $\varphi = \iota \circ \overline{\varphi} \circ \pi$.*

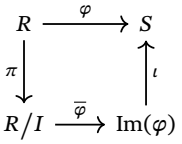


Figure 6.1: The commutative diagram arising from the First Isomorphism Theorem 6.1.1.

Proof. The Induced Map Lemma, with $J := \langle 0_S \rangle$, shows that $\overline{\varphi}$ is a well-defined ring homomorphism. For every $r \in R$, the coset $r + I$ lies in $\operatorname{Ker}(\overline{\varphi})$ if and only if $0_S = \overline{\varphi}(r + I) = \varphi(r)$, which means $r \in \operatorname{Ker}(\varphi)$. Since $\operatorname{Ker}(\overline{\varphi}) = \langle 0 + I \rangle$, Corollary 5.1.12 shows that $\overline{\varphi}$ is injective. By definition, the map $\overline{\varphi}$ surjects onto $\operatorname{Im}(\varphi)$, so Proposition 6.0.2 implies that $\overline{\varphi}$ is a ring isomorphism. The second part follows from the definition of $\overline{\varphi}$. □

6.1.2 Problem. Confirm that $\mathbb{Z}[\mathrm{i}]/\langle 1 + 3\,\mathrm{i} \rangle \cong \mathbb{Z}/\langle 10 \rangle$.

Solution. By Problem 5.0.4, there is a unique ring homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}[\mathrm{i}]/\langle 1 + 3\,\mathrm{i} \rangle$. As

$$\mathrm{i} + \langle 1 + 3\,\mathrm{i} \rangle = (-1 + \langle 1 + 3\,\mathrm{i} \rangle)(-\mathrm{i} + \langle 1 + 3\,\mathrm{i} \rangle) = (3\,\mathrm{i} + \langle 1 + 3\,\mathrm{i} \rangle)(-\mathrm{i} + \langle 1 + 3\,\mathrm{i} \rangle) = 3 + \langle 1 + 3\,\mathrm{i} \rangle\,,$$

it follows that $(a+b\,\mathrm{i})+\langle 1 + 3\,\mathrm{i} \rangle = (a+3b)+\langle 1 + 3\,\mathrm{i} \rangle$, so φ is surjective. For every $m \in \operatorname{Ker}(\varphi)$, we have $m \in \langle 1 + 3\,\mathrm{i} \rangle$. Hence, there exist $c, d \in \mathbb{Z}$ such that $m = (c + d\,\mathrm{i})(1 + 3\,\mathrm{i}) = (c - 3d) + (3c + d)\mathrm{i}$, so $3c = -d$, $m = c + 3(-d) = 10c$, and $\operatorname{Ker}(\varphi) \subseteq \langle 10 \rangle$. Conversely, we have $3^2 + \langle 1 + 3\,\mathrm{i} \rangle = (\mathrm{i} + \langle 1 + 3\,\mathrm{i} \rangle)^2 = \mathrm{i}^2 + \langle 1 + 3\,\mathrm{i} \rangle = -1 + \langle 1 + 3\,\mathrm{i} \rangle$ and $10 + \langle 1 + 3\,\mathrm{i} \rangle = 0 + \langle 1 + 3\,\mathrm{i} \rangle$, so $\langle 10 \rangle \subseteq \operatorname{Ker}(\varphi)$. Thus, the First Isomorphism Theorem yields $\mathbb{Z}/\langle 10 \rangle \cong \mathbb{Z}[\mathrm{i}]/\langle 1 + 3\,\mathrm{i} \rangle$. □

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6.1.3 Problem. Prove that the quotient $\mathbb{C}[x,y]/\langle x\,y\rangle$ is isomorphic to the subring of the product $\mathbb{C}[x]\times\mathbb{C}[y]$ consisting of (f,g) such that $\mathrm{ev}_{x=0}(f) = f(0) = g(0) = \mathrm{ev}_{y=0}(g)$.

Solution. Let $R \subseteq \mathbb{C}[x]\times\mathbb{C}[y]$ be the subring. Consider the map $\varphi\colon \mathbb{C}[x,y] \to \mathbb{C}[x]\times\mathbb{C}[y]$ defined by $h(x,y) \mapsto (h(x,0), h(0,y))$. As each component is an evaluation map, φ is a ring homomorphism. The kernel of the map $\mathrm{ev}_{y=0}$ is $\langle y\rangle$ and the kernel of the map $\mathrm{ev}_{x=0}$ is $\langle x\rangle$, so $\operatorname{Ker}\varphi = \langle x\rangle \cap \langle y\rangle = \langle x\,y\rangle$. For every $h \in \mathbb{C}[x,y]$, both components of $\varphi(h)$ take the value $h(0,0)$ at 0, so $\operatorname{Im}\varphi \subseteq R$. Conversely, given $(f,g) \in R$, set $h(x,y) \coloneqq f(x) + g(y) - f(0)$. It follows that

$$\begin{aligned} h(x,0) &= f(x) + g(0) - f(0) = f(x) && \text{and} \\ h(0,y) &= f(0) + g(y) - f(0) = g(y)\,, && \text{so } \operatorname{Im}\varphi = R. \end{aligned}$$

The First Isomorphism Theorem gives $\mathbb{C}[x,y]/\langle x\,y\rangle \cong R$. □

Clearly $\langle x\,y\rangle \subseteq \langle x\rangle \cap \langle y\rangle$. If $h(x,0) = 0$, then every monomial in h is divisible by y, and, if $h(0,y) = 0$, then every monomial in h is divisible by x. Hence, every monomial is divisible by $x\,y$, so $h \in \langle x\,y\rangle$. Thus, we deduce that $\operatorname{Ker}\varphi = \langle x\,y\rangle$.

Aside from the First Isomorphism Theorem 6.1.1, there are almost no methods for recognizing a quotient ring, because it will usually not be a familiar ring.

Quotients arising from an ideal and a subring are isomorphic.

6.1.4 Second Isomorphism Theorem. *Let R be a ring. For every ideal I in R and every subring S of R, the sum $S+I$ is a subring of R, the set I is an ideal in the subring $S+I$, the intersection $S\cap I$ is an ideal in S, and the map $\overline{\eta}\colon S/(S\cap I) \to (S+I)/I$, defined by $\overline{\eta}(s+(S\cap I)) = s+I$ for all $s\in S$, is a ring isomorphism.*

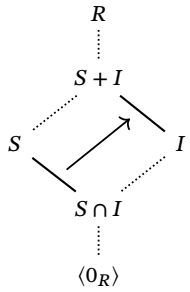


Figure 6.2: The Hasse diagram arising from the Second Isomorphism Theorem 6.1.4

Proof. We claim $S+I \coloneqq \{s+g \in R \mid s\in S \text{ and } g\in I\}$ is a subring. For all $r,s\in S$ and all $f,g\in I$, we have

$$\begin{aligned} (r+f)-(s+g) &= (r-s)+(f-g) \in S+I \\ (r+f)(s+g) &= r\,s + r\,g + f\,s + f\,g \in S+I \end{aligned}$$

because $r-s, r\,s \in S$ and $f-g, r\,g+f\,s+f\ \in I$. As $1_R \in S$ and $0_R \in I$, we also have $1_R = 1_R + 0_R \in S+I$, so $S+I$ is a subring.

Since I is an ideal in R and $S+I$ is a subring of R, the subset I is an ideal in $S+I$. We claim that $S\cap I$ is an ideal in S. For all $f,g\in S\cap I$ and any $s\in S$, the elements $f-g$, $s\,f$, and $f\,s$ all belong to both I and S. As these elements belong to $S\cap I$, we deduce that $S\cap I$ is an ideal in the subring S.

Consider the map $\eta\colon S \to (S+I)/I$ defined by $\eta(s) \coloneqq s+I$ for all $s\in S$. For all $r,s\in S$, we have $\eta(r+s) = (r+s)+I = (r+I)+(s+I)$, $\eta(r\,s) = (r\,s)+I = (r+I)(s+I)$, and $\eta(1_S) = 1_R + I$, so η is a ring homomorphism. From the definition of $S+I$, we see that η is surjective. We claim that $\operatorname{Ker}(\eta) = S\cap I$.

$\subseteq$: Suppose that $f \in \operatorname{Ker}(\eta)$. We have $f+I = \eta(f) = 0+I = I$, so $f\in I$. As $\operatorname{Ker}(\eta)\subseteq S$, the membership $f\in S$ implies that $f\in S\cap I$ and $\operatorname{Ker}(\eta)\subseteq S\cap I$.

$\supseteq$: Suppose that $f\in S\cap I$. As $f\in I$, we have $\eta(f) = f+I = I$, so $f\in\operatorname{Ker}(\eta)$ and $\operatorname{Ker}(\eta)\supseteq S\cap I$.

Since $\operatorname{Ker}(\eta) = S\cap I$ and η is surjective, the First Isomorphism Theorem proves that $\overline{\eta}\colon S/(S\cap I) \to (S+I)/I$ is a ring isomorphism. □

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6.2 Ideals in a Quotient Ring

How are ideals in a quotient ring related to ideals in its ambient ring? Every surjective ring homomorphism gives rise to a bijection between certain ideals in its source and all ideals in its target.

6.2.0 Correspondence Theorem. *Assume that $\varphi\colon R \to S$ is a ring homomorphism with kernel $K := \operatorname{Ker}(\varphi)$.*

(pullback) *For each ideal J in S, the preimage $\varphi^{-1}(J) := \{r \mid \varphi(r) \in J\}$ is an ideal in R containing the ideal K.*

(injection) *For each ideal J in S, the composition of the map $\varphi\colon R \to S$ and the canonical surjection $\pi\colon S \to S/J$ induces an injective ring homomorphism $\overline{\pi \circ \varphi}\colon R/\varphi^{-1}(J) \to S/J$. When φ is surjective, the map $\overline{\pi \circ \varphi}$ is a ring isomorphism.*

(bijection) *Assume that the map φ is surjective. The maps $I \mapsto \varphi(I)$ and $J \mapsto \varphi^{-1}(J)$ are inverse inclusion-preserving bijections between the set of ideals in R containing the kernel K and the set of ideals in S.*

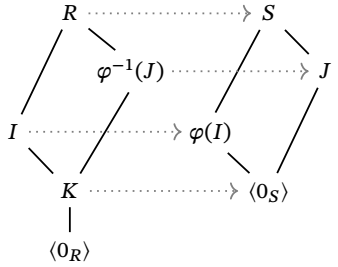


Figure 6.3: The Hasse diagram of ideals in the Correspondence Theorem

Proof. For all $f, g \in \varphi^{-1}(J)$ and all $r \in R$, the ring homomorphism gives $\varphi(f+g) = \varphi(f) + \varphi(g)$, $\varphi(rf) = \varphi(r)\varphi(f)$, $\varphi(fr) = \varphi(f)\varphi(r)$ which shows that all these elements belong to the ideal J, so $f+g$, rf, and fr belong to $\varphi^{-1}(J)$. Hence, the preimage $\varphi^{-1}(J)$ is an ideal in R. As $0_S \in J$ and $\varphi^{-1}(0_S) = K$, the preimage $\varphi^{-1}(J)$ contains K.

By the Induced Map Lemma, the map $\overline{\pi \circ \varphi}\colon R/\varphi^{-1}(J) \to S/J$ is a well-defined ring homomorphism. As $\operatorname{Ker}(\pi) = J$, it follows that $\operatorname{Ker}(\pi \circ \varphi) = \varphi^{-1}(J)$ and $\operatorname{Ker}(\overline{\pi \circ \varphi}) = \langle 0 + \varphi^{-1}(J) \rangle$, so this induced map is also injective. When φ is surjective, the composition $\pi \circ \varphi$ is also surjective, whence $\overline{\pi \circ \varphi}$ is surjective.

Assume that the map φ is surjective. We claim that, for each ideal I in R, the image $\varphi(I)$ is an ideal in S. For all $p, q \in \varphi(I)$ and all $s \in S$, there exist $f, g \in I$ and $r \in R$ such that $\varphi(f) = p$, $\varphi(g) = q$, and $\varphi(r) = s$. It follows that the elements $p+q = \varphi(f)+\varphi(g) = \varphi(f+g)$, $sp = \varphi(r)\varphi(f) = \varphi(rf)$, and $ps = \varphi(f)\varphi(r) = \varphi(fr)$ belong to the image $\varphi(I)$, so $\varphi(I)$ is an ideal in S. Since $\varphi(K) = \langle 0_S \rangle$, the image of any ideal contained in the kernel K is zero ideal in S.

The preceding paragraph shows that the map $I \mapsto \varphi(I)$ sends each ideal I in R containing K to an ideal in S. Conversely, the first part shows that the map $J \mapsto \varphi^{-1}(J)$ sends each ideal J in S to an ideal in R containing K. We show that these maps compose in either order to the identity. For each ideal J in S, the inclusion $\varphi(\varphi^{-1}(J)) \subseteq J$ holds for any map, and the surjectivity of φ gives the reverse inclusion: each $s \in J$ equals $\varphi(r)$ for some $r \in R$, and $r \in \varphi^{-1}(J)$. For each ideal I in R containing K, we have $I \subseteq \varphi^{-1}(\varphi(I))$. Conversely, if $r \in \varphi^{-1}(\varphi(I))$, then $\varphi(r) = \varphi(f)$ for some $f \in I$, so $r - f \in K \subseteq I$ and hence $r = (r-f) + f \in I$. Finally, if $I \subseteq I'$, then $\varphi(I) \subseteq \varphi(I')$, and if $J \subseteq J'$, then $\varphi^{-1}(J) \subseteq \varphi^{-1}(J')$, so both maps preserve inclusions. □

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6.2.1 Problem. Describe all ideals in the ring $\mathbb{Z}/\langle 6\rangle$.

Compare with Problem 5.1.9.

Solution. Given the inclusions $\langle 6\rangle \subset \langle 2\rangle$ and $\langle 6\rangle \subset \langle 3\rangle$, the Induced Map Lemma shows that the identity map $\mathrm{id}_{\mathbb{Z}} \colon \mathbb{Z} \to \mathbb{Z}$ induces ring homomorphisms $\varphi \colon \mathbb{Z}/\langle 6\rangle \to \mathbb{Z}/\langle 2\rangle$ and $\psi \colon \mathbb{Z}/\langle 6\rangle \to \mathbb{Z}/\langle 3\rangle$ where $\operatorname{Ker}(\varphi) = \langle [2]_6\rangle$ and $\operatorname{Ker}(\psi) = \langle [3]_6\rangle$. As $\mathbb{Z}/\langle 2\rangle$ and $\mathbb{Z}/\langle 3\rangle$ are both fields, they contain only two distinct ideals: the zero ideal and the whole ring. The Correspondence Theorem shows that $\langle [2]_6\rangle$ and $\langle [1]_6\rangle$ are the ideals in $\mathbb{Z}/\langle 6\rangle$ containing $\langle [2]_6\rangle$, and $\langle [3]_6\rangle$ and $\langle [1]_6\rangle$ are the ideals in $\mathbb{Z}/\langle 6\rangle$ containing $\langle [3]_6\rangle$. Since $[4]_6\,[2]_6 = [2]_6$ and $[5]_6$ is a unit, we conclude that the four ideals in $\mathbb{Z}/\langle 6\rangle$ are $\langle [0]_6\rangle$, $\langle [2]_6\rangle$, $\langle [3]_6\rangle$, and $\langle [1]_6\rangle$. □

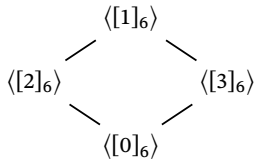


Figure 6.4: Hasse diagram of ideals in the quotient ring $\mathbb{Z}/\langle 6\rangle$

6.2.2 Problem. For every $m \in \mathbb{N}$, describe all ideals in the quotient ring $\mathbb{C}[x]/\langle x^{m+1}\rangle$.

Solution. For every $m \in \mathbb{N}$, let $I_m \coloneqq \langle x^{m+1}\rangle$ be an ideal in $\mathbb{C}[x]$. We prove, by induction on m, that the $m+2$ ideals in $\mathbb{C}[x]/I_m$ are

$$\langle 0 + I_m\rangle, \langle x^m + I_m\rangle, \langle x^{m-1} + I_m\rangle, \dots, \langle x + I_m\rangle, \langle 1 + I_m\rangle \,.$$

The First Isomorphism Theorem gives $\mathbb{C}[x]/\langle x\rangle \cong \mathbb{C}$. Since $\mathbb{C}$ is a field, the only ideals are $\langle 0\rangle$ and $\langle 1\rangle$, which gives the base case $m = 0$. Assume that $k \in \mathbb{N}$ satisfies $k \geqslant 1$. Given $I_k \subset I_{k-1}$, the Induced Map Lemma shows that the map $\mathrm{id}_{\mathbb{C}[x]}$ induces a ring homomorphism $\varphi \colon \mathbb{C}[x]/I_k \to \mathbb{C}[x]/I_{k-1}$ where $\operatorname{Ker}(\varphi) = \langle x^{k-1} + I_k\rangle$. Hence, the induction hypothesis and the Correspondence Theorem show that the $k+1$ ideals in the ring $\mathbb{C}[x]/I_k$ containing $\langle x^k + I_k\rangle$ are

$$\begin{aligned} \varphi^{-1}(\langle 0 + I_{k-1}\rangle) &= \langle x^k + I_k\rangle \\ \varphi^{-1}(\langle x^{k-1} + I_{k-1}\rangle) &= \langle x^{k-1} + I_k\rangle \\ &\vdots \\ \varphi^{-1}(\langle 1 + I_{k-1}\rangle) &= \langle 1 + I_k\rangle \,. \end{aligned}$$

It remains to show that $\langle 0 + I_k\rangle$ is the only other ideal in the ring $\mathbb{C}[x]/I_k$. Thus, it suffices to prove that any nonzero ideal contains the ideal $\langle x^k + I_k\rangle$. A nonzero polynomial f in $\mathbb{C}[x]$ having degree at most k has the form $f = a_k\,x^k + a_{k+1}\,x^{k-1} + \cdots + a_\ell\,x^\ell$ where $\ell \in \mathbb{N}$, $a_k, a_{k-1}, \dots, a_\ell \in \mathbb{C}$, and $a_\ell \neq 0$. As $(a_\ell^{-1}\,x^{k-\ell}\,f) \,\%\, x^{k+1} = x^k$, we have the required inclusion $\langle x^k + I_k\rangle \subseteq \langle f + I_k\rangle$. □

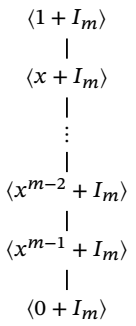


Figure 6.5: Hasse diagram of ideals in the ring $\mathbb{C}[x]/I_m$

One can show that every element in $\mathbb{C}[x]/I_m$ is either nilpotent or a unit.

The quotient rings arising from nested ideals in a ring are also related by an isomorphism.

6.2.3 Third Isomorphism Theorem. *Let I be an ideal in the ring R.*

(ideals) *Every ideal in R/I has the form $J/I \coloneqq \{r + I \mid r \in J\}$ for some ideal J in R containing I.*

(isomorphism) *For each ideal J in R containing I, the composition of the quotient maps $R \to R/I$ and $R/I \to (R\,/\,I)/(J\,/\,I)$ induces a ring isomorphism $R/J \to (R\,/\,I)/(J\,/\,I)$.*

Proof. Applying the Correspondence Theorem to the surjective ring homomorphism $\pi \colon R \to R/I$ establishes both parts. □

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6.4 Exercises

6.4.1 Problem. Determine which of the following statements are true. If a statement is false, provide a counterexample and, if possible, salvage it by modifying the statement so that it becomes true.

- **i.** Isomorphic rings have the same number of elements.
- **ii.** Any two rings with the same number of elements are isomorphic.
- **iii.** When two rings are isomorphic, then one is a field if and only if the other is.
- **iv.** The isomorphism $R/I \cong R/J$ implies that $I = J$.
- **v.** Every ring homomorphism factors uniquely through the quotient by its kernel.
- **vi.** For all nested ideal $I \subseteq J$ in a ring R, the quotient J/I is always a subring of R/I.
- **vii.** Distinct ideals of R always correspond to distinct ideals of R/I.
- **viii.** For all ideals I and J of a ring R, we have $R/(I \cap J) \cong R/I \times R/J$.

6.4.2 Problem. Let R be a ring and let u be a unit in R. Confirm that the map $\varphi : R \to R$ defined, for any element a in the ring R, by $\varphi(a) = u\,a\,u^{-1}$ is an automorphism.

6.4.3 Problem. Show that $\mathbb{Z}/\langle 4\rangle$ is not isomorphic to $\mathbb{Z}/\langle 2\rangle \times \mathbb{Z}/\langle 2\rangle$.

6.4.4 Problem (2-out-of-6 property)**.** Let $\theta : Q \to R$, $\varphi : R \to S$, and $\psi : S \to T$ be ring homomorphisms. When the compositions $\varphi\,\theta$ and $\psi\,\varphi$ are ring isomorphisms, prove that θ, φ, θ, and $\psi\,\varphi\,\theta$ are also ring isomorphisms.

6.4.5 Problem. Each quotient ring R/I in the left column of Table 6.1 is isomorphic to a ring S in the right column. Match each quotient ring with its isomorphic partner and prove that they are isomorphic be describing a surjective ring homomorphism $\varphi : R \to S$ with kernel I.

$\frac{R}{I}$	$\frac{\mathbb{Z}[x]}{\langle 35, 80, x\rangle}$	$\frac{\mathbb{Z}[x]}{\langle x^2+10x+26\rangle}$	$\frac{\mathbb{R}[x]}{\langle x^2-5\rangle}$	$\frac{\mathbb{R}[x]}{\langle x^2+5\rangle}$	$\frac{\mathbb{C}[x]}{\langle x^2\rangle}$	$\frac{\mathbb{Q}[x]}{\langle x^2-5\rangle}$

S	$\mathbb{Z}$	$\frac{\mathbb{Z}}{\langle 5\rangle}$	$\frac{\mathbb{Z}}{\langle 25\rangle}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}\times\mathbb{R}$	$\mathbb{Z}[\mathrm{i}]$	$\mathbb{Q}[x]$	$\mathbb{Q}[\sqrt{5}]$	$\mathbb{R}[x]$	$\left\{\begin{bmatrix} z & w \\ 0 & z \end{bmatrix} \middle\vert\, z, w \in \mathbb{C}\right\}$

Table 6.1: Table of quotient rings and rings

6.4.6 Problem.

- **i.** Prove that $\mathbb{Z}/\langle 60\rangle$ is isomorphic to $\mathbb{Z}/\langle 3\rangle \times \mathbb{Z}/\langle 4\rangle \times \mathbb{Z}/\langle 5\rangle$.
- **ii.** Exhibit elements e_1, e_2, and e_3 in $\mathbb{Z}/\langle 60\rangle$ such that

$$e_1^2 = e_1 \qquad e_2^2 = e_2 \qquad e_3^2 = e_3 \qquad e_2\,e_3 = 0 \qquad e_1\,e_3 = 0 \qquad e_1\,e_2 = 0$$

and $[1]_{60} = e_1 + e_2 + e_3$.

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