Problems 09

Due: Friday, 13 November 2026 before 23:59 ET

P9.1. Let $\theta\colon Q \to R$, $\varphi\colon R \to S$, and $\psi\colon S \to T$ be ring homomorphisms. When the composite maps $\varphi \circ \theta\colon Q \to S$ and $\psi \circ \varphi\colon R \to T$ are ring isomorphisms, prove that θ, φ, ψ, and $\psi \circ \varphi \circ \theta$ are also ring isomorphisms.

P9.2. **i.** Express the ring $\mathbb{Z}/\langle 600\rangle$ as a product of three quotient rings.

ii. Exhibit elements e_0, e_1, and e_2 in $\mathbb{Z}/\langle 600\rangle$ such that

$$e_0^2 = e_0\,, \qquad e_1^2 = e_1\,, \qquad e_2^2 = e_2\,, \qquad e_1\,e_2 = 0\,, \qquad e_0\,e_2 = 0\,, \qquad e_0\,e_1 = 0\,,$$

and $[1]_{600} = e_0 + e_1 + e_2$.

P9.3. **i.** Let $\varphi\colon R \to S$ be a ring homomorphism between commutative rings. Consider multiplicative subsets $D \subset R$ and $E \subset S$ that satisfy $\varphi(D) \subseteq E$. Prove that there exists a unique ring homomorphism $\widehat{\varphi}\colon R[D^{-1}] \to S[E^{-1}]$ such that $\widehat{\varphi}(r/1) = \varphi(r)/1$ for all $r \in R$.

ii. Show that every automorphism of a domain admits a unique extension to its field of fractions that is also an automorphism.

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